Integrals

Definitions

Definite Integral : Suppose $f(x)$ is continuous on $[a,b]$. Divide $[a,b]$ into n subintervals of width Δx and choose x_i^* from each interval. Then $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$.

Anti-Derivative : An anti-derivative of $f(x)$ is a function, $F(x)$, such that $F'(x) = f(x)$.

Indefinite Integral : $\int f(x)\,dx = F(x) + c$ where $F(x)$ is an anti-derivative of $f(x)$.

Fundamental Theorem of Calculus

Part I : If $f(x)$ is continuous on $[a,b]$ then $g(x) = \int_a^x f(t)\,dt$ is also continuous on $[a,b]$ and $g'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$.

Part II : $f(x)$ is continuous on $[a,b]$, $F(x)$ is an anti-derivative of $f(x)$, *i.e.* $F(x) = \int f(x)\,dx$, then $\int_a^b f(x)\,dx = F(b) - F(a)$.

Variants of Part I :

$$\frac{d}{dx}\int_a^{u(x)} f(t)\,dt = u'(x)\,f\left[u(x)\right]$$

$$\frac{d}{dx}\int_{v(x)}^{b} f(t)\,dt = -v'(x)\,f\left[v(x)\right]$$

$$\frac{d}{dx}\int_{v(x)}^{u(x)} f(t)\,dt = u'(x)\,f\left[u(x)\right] - v'(x)\,f\left[v(x)\right]$$

Properties

$\int f(x) \pm g(x)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

$\int c f(x)\,dx = c\int f(x)\,dx$, c is a constant

$\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

$\int_a^b c f(x)\,dx = c\int_a^b f(x)\,dx$, c is a constant

$\int_a^a f(x)\,dx = 0$

$\int_a^b c\,dx = c(b-a)$, c is a constant

$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

$\left|\int_a^b f(x)\,dx\right| \le \int_a^b \left|f(x)\right|dx$

$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ for any value c.

If $f(x) \ge g(x)$ on $a \le x \le b$ then $\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$

If $f(x) \ge 0$ on $a \le x \le b$ then $\int_a^b f(x)\,dx \ge 0$

If $m \le f(x) \le M$ on $a \le x \le b$ then $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$

Common Integrals

$\int k\,dx = k\,x + c$

$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c,\ n \ne -1$

$\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln|x| + c$

$\int \mathbf{e}^u\,du = \mathbf{e}^u + c$

$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + c$

$\int \ln(u)\,du = u\ln(u) - u + c$

$\int \cos(u)\,du = \sin(u) + c$

$\int \sec(u)\tan(u)\,du = \sec(u) + c$

$\int \tan(u)\,du = \ln|\sec(u)| + c$

$\int \sin(u)\,du = -\cos(u) + c$

$\int \csc(u)\cot(u)du = -\csc(u) + c$

$\int \tan(u)\,du = -\ln|\cos(u)| + c$

$\int \sec^2(u)\,du = \tan(u) + c$

$\int \sec(u)du = \ln\left|\sec(u) + \tan(u)\right| + c$

$\int \frac{1}{a^2+u^2}\,du = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + c$

$\int \csc^2(u)\,du = -\cot(u) + c$

$\int \csc(u)du = -\ln\left|\csc(u) + \cot(u)\right| + c$

$\int \frac{1}{\sqrt{a^2-u^2}}\,du = \sin^{-1}\left(\frac{u}{a}\right) + c$

Standard Integration Techniques

***u* Substitution :** $\int_a^b f(g(x))\,g'(x)\,dx$ will convert the integral into $\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$ using the substitution $u = g(x)$ where $du = g'(x)dx$. For indefinite integrals drop the limits of integration.

Example $\int_1^2 5x^2\cos\left(x^3\right)\,dx$

$u = x^3 \quad\Rightarrow\quad du = 3x^2dx \quad\Rightarrow\quad x^2dx = \frac{1}{3}du$

$x = 1 \quad\Rightarrow\quad u = 1^3 = 1 \quad :: \quad x = 2 \quad\Rightarrow\quad u = 2^3 = 8$

$$\int_1^2 5x^2\cos\left(x^3\right)\,dx = \int_1^8 \frac{5}{3}\cos(u)\,du = \frac{5}{3}\sin(u)\Big|_1^8 = \frac{5}{3}\Big(\sin(8) - \sin(1)\Big)$$

Products and (some) Quotients of Trig Functions

For $\int \sin^n(x)\cos^m(x)\,dx$ we have the following :

1. ***n* odd.** Strip 1 sine out and convert rest to cosines using $\sin^2(x) = 1 - \cos^2(x)$, then use the substitution $u = \cos(x)$.
2. ***m* odd.** Strip 1 cosine out and convert rest to sines using $\cos^2(x) = 1 - \sin^2(x)$, then use the substitution $u = \sin(x)$.
3. ***n* and *m* both odd.** Use either 1. or 2.
4. ***n* and *m* both even.** Use double angle and/or half angle formulas to reduce the integral into a form that can be integrated.

For $\int \tan^n(x)\sec^m(x)\,dx$ we have the following :

1. ***n* odd.** Strip 1 tangent and 1 secant out and convert the rest to secants using $\tan^2(x) = \sec^2(x) - 1$, then use the substitution $u = \sec(x)$.
2. ***m* even.** Strip 2 secants out and convert rest to tangents using $\sec^2(x) = 1 + \tan^2(x)$, then use the substitution $u = \tan(x)$.
3. ***n* odd and *m* even.** Use either 1. or 2.
4. ***n* even and *m* odd**. Each integral will be dealt with differently.

Trig Formulas : $\sin(2x) = 2\sin(x)\cos(x)$, $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$, $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$

Example $\int \tan^3(x)\sec^5(x)\,dx$

$$\begin{aligned}\int \tan^3 x\ \sec^5 x\,dx &= \int \tan^2 x\ \sec^4 x\ \tan x\ \sec x\,dx\\ &= \int \left(\sec^2(x) - 1\right)\sec^4(x)\ \tan(x)\sec(x)dx\\ &= \int \left(u^2 - 1\right)u^4du \qquad \left[u = \sec(x)\right]\\ &= \tfrac{1}{7}\sec^7(x) - \tfrac{1}{5}\sec^5(x) + c\end{aligned}$$

Example $\int \frac{\sin^5(x)}{\cos^3(x)}\,dx$

$$\begin{aligned}\int \frac{\sin^5 x}{\cos^3 x}\,dx &= \int \frac{\sin^4 x\sin x}{\cos^3 x}\,dx = \int \frac{(\sin^2 x)^2\sin x}{\cos^3 x}\,dx\\ &= \int \frac{(1-\cos^2(x))^2\sin(x)}{\cos^3(x)}\,dx \qquad \left[u = \cos(x)\right]\\ &= -\int \frac{(1-u^2)^2}{u^3}du = -\int \frac{1-2u^2+u^4}{u^3}du\\ &= \tfrac{1}{2}\sec^2(x) + 2\ln\left|\cos(x)\right| - \tfrac{1}{2}\cos^2(x) + c\end{aligned}$$

Integration by Parts : $\int u\,dv = uv - \int v\,du$ and $\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$. Choose u and dv from integral and compute du by differentiating u and compute v using $v = \int dv$.

Example $\int x\mathbf{e}^{-x}\,dx$

$u = x \quad dv = \mathbf{e}^{-x} \quad \Rightarrow \quad du = dx \quad v = -\mathbf{e}^{-x}$

$$\int x\mathbf{e}^{-x}\,dx = -x\mathbf{e}^{-x} + \int \mathbf{e}^{-x}\,dx = -x\mathbf{e}^{-x} - \mathbf{e}^{-x} + c$$

Example $\int_3^5 \ln(x)\,dx$

$u = \ln(x) \quad dv = dx \quad \Rightarrow \quad du = \frac{1}{x}dx \quad v = x$

$$\int_3^5 \ln(x)\,dx = x\ln(x)\Big|_3^5 - \int_3^5 dx = (x\ln(x) - x)\Big|_3^5 = 5\ln(5) - 3\ln(3) - 2$$

Trig Substitutions : If the integral contains the following root use the given substitution and formula to convert into an integral involving trig functions.

$\sqrt{a^2 - b^2x^2} \Rightarrow x = \frac{a}{b}\sin(\theta)$	$\sqrt{b^2x^2 - a^2} \Rightarrow x = \frac{a}{b}\sec(\theta)$	$\sqrt{a^2 + b^2x^2} \Rightarrow x = \frac{a}{b}\tan\theta$
$\cos^2(\theta) = 1 - \sin^2(\theta)$	$\tan^2(\theta) = \sec^2(\theta) - 1$	$\sec^2(\theta) = 1 + \tan^2(\theta)$

Example $\int \frac{16}{x^2\sqrt{4 - 9x^2}}\,dx$

$x = \frac{2}{3}\sin(\theta) \quad \Rightarrow \quad dx = \frac{2}{3}\cos(\theta)\,d\theta$

$\sqrt{4 - 9x^2} = \sqrt{4 - 4\sin^2(\theta)} = \sqrt{4\cos^2(\theta)} = 2\,|\cos(\theta)|$

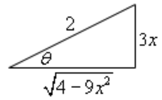
Recall $\sqrt{x^2} = |x|$. Because we have an indefinite integral we'll assume positive and drop absolute value bars. If we had a definite integral we'd need to compute θ's and remove absolute value bars based on that and,

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

In this case we have $\sqrt{4 - 9x^2} = 2\cos(\theta)$.

$$\int \frac{16}{\frac{4}{9}\sin^2(\theta)(2\cos(\theta))}\left(\tfrac{2}{3}\cos\theta\right)d\theta = \int \frac{12}{\sin^2(\theta)}\,d\theta = \int 12\csc^2(\theta)\,d\theta = -12\cot(\theta) + c$$

Use Right Triangle Trig to go back to x's. From substitution we have $\sin(\theta) = \frac{3x}{2}$ so,



From this we see that $\cot(\theta) = \frac{\sqrt{4 - 9x^2}}{3x}$. So,

$$\int \frac{16}{x^2\sqrt{4 - 9x^2}}\,dx = -\frac{4\sqrt{4 - 9x^2}}{x} + c$$

Partial Fractions : If integrating a rational expression involving polynomials, $\int \frac{P(x)}{Q(x)}\,dx$, where the degree of $P(x)$ is smaller than the degree of $Q(x)$. Factor denominator as completely as possible and find the partial fraction decomposition of the rational expression. Integrate the partial fraction decomposition (P.F.D.). For each factor in the denominator we get term(s) in the decomposition according to the following table.

Factor of $Q(x)$	Term in P.F.D	Factor is $Q(x)$	Term in P.F.D
$ax + b$	$\frac{A}{ax + b}$	$(ax + b)^k$	$\frac{A_1}{ax + b} + \frac{A_2}{(ax + b)^2} + \cdots + \frac{A_k}{(ax + b)^k}$
$ax^2 + bx + c$	$\frac{Ax + B}{ax^2 + bx + c}$	$(ax^2 + bx + c)^k$	$\frac{A_1x + B_1}{ax^2 + bx + c} + \cdots + \frac{A_kx + B_k}{(ax^2 + bx + c)^k}$

Example $\int \frac{7x^2 + 13x}{(x - 1)(x^2 + 4)}\,dx$

$$\int \frac{7x^2 + 13x}{(x - 1)(x^2 + 4)}\,dx = \int \frac{4}{x - 1} + \frac{3x + 16}{x^2 + 4}\,dx = \int \frac{4}{x - 1} + \frac{3x}{x^2 + 4} + \frac{16}{x^2 + 4}\,dx$$

$$= 4\ln|x - 1| + \tfrac{3}{2}\ln\left(x^2 + 4\right) + 8\tan^{-1}\left(\tfrac{x}{2}\right)$$

Here is partial fraction form and recombined.

$$\frac{7x^2 + 13x}{(x-1)(x^2+4)} = \frac{A}{x-1} + \frac{Bx + C}{x^2 + 4} = \frac{A(x^2 + 4) + (Bx + C)(x - 1)}{(x-1)(x^2+4)}$$

Set numerators equal and collect like terms.

$$7x^2 + 13x = (A + B)x^2 + (C - B)x + 4A - C$$

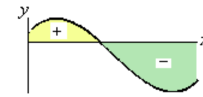
Set coefficients equal to get a system and solve to get constants.

$A + B = 7 \quad C - B = 13 \quad 4A - C = 0$

$A = 4 \quad B = 3 \quad C = 16$

Applications of Integrals

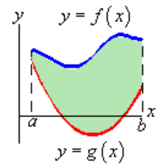
Net Area : $\int_a^b f(x)\,dx$ represents the net area between $f(x)$ and the x-axis with area above x-axis positive and area below x-axis negative.



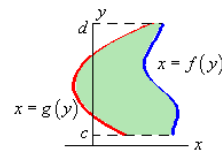
Area Between Curves : The general formulas for the two main cases for each are,

$y = f(x) \Rightarrow A = \int_a^b$ [upper function] − [lower function] dx & $x = f(y) \Rightarrow A = \int_c^d$ [right function] − [left function] dy

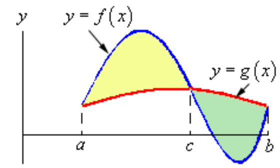
If the curves intersect then the area of each portion must be found individually. Here are some sketches of a couple possible situations and formulas for a couple of possible cases.



$A = \int_a^b f(x) - g(x)\,dx$



$A = \int_c^d f(y) - g(y)\,dy$



$A = \int_a^c f(x) - g(x)\,dx + \int_c^b g(x) - f(x)\,dx$

Volumes of Revolution : The two main formulas are $V = \int A(x)\,dx$ and $V = \int A(y)\,dy$. Here is some general information about each method of computing and some examples.

Rings

$A = \pi\left((\text{outer radius})^2 - (\text{inner radius})^2\right)$

Limits: x/y of right/bot ring to x/y of left/top ring

Horz. Axis use $f(x)$, $g(x)$, $A(x)$ and dx.

Vert. Axis use $f(y)$, $g(y)$, $A(y)$ and dy.

Cylinders/Shells

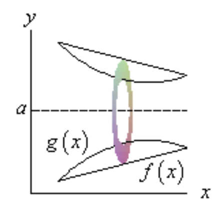
$A = 2\pi(\text{radius})(\text{width / height})$

Limits : x/y of inner cyl. to x/y of outer cyl.

Horz. Axis use $f(y)$, $g(y)$, $A(y)$ and dy.

Vert. Axis use $f(x)$, $g(x)$, $A(x)$ and dx.

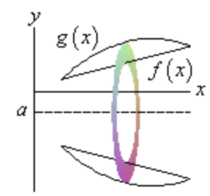
Ex. Axis : $y = a > 0$



outer radius : $a - f(x)$

inner radius : $a - g(x)$

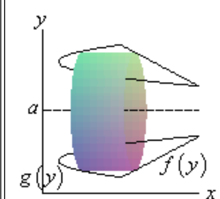
Ex. Axis : $y = a \leq 0$



outer radius: $|a| + g(x)$

inner radius: $|a| + f(x)$

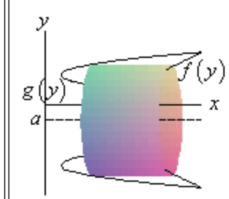
Ex. Axis : $y = a > 0$



radius : $a - y$

width : $f(y) - g(y)$

Ex. Axis : $y = a \leq 0$



radius : $|a| + y$

width : $f(y) - g(y)$

These are only a few cases for horizontal axis of rotation. If the axis of rotation is the x-axis use the $y = a \leq 0$ case with $a = 0$. For vertical axis of rotation ($x = a > 0$ and $x = a \leq 0$) interchange x and y to get appropriate formulas.

Work : If a force of $F(x)$ moves an object in $a \leq x \leq b$, the work done is $W = \int_a^b F(x)\,dx$

Average Function Value : The average value of $f(x)$ on $a \leq x \leq b$ is $f_{avg} = \frac{1}{b - a}\int_a^b f(x)\,dx$

Arc Length & Surface Area : The three basic formulas are,

$L = \int_a^b ds \qquad SA = \int_a^b 2\pi y \, ds$ (rotate about x-axis) $\qquad SA = \int_a^b 2\pi x \, ds$ (rotate about y-axis)

where ds is dependent upon the form of the function being worked with as follows.

$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$ if $y = f(x), \;\; a \le x \le b \qquad ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$ if $x = f(t), y = g(t), \;\; a \le t \le b$

$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy$ if $x = f(y), \; a \le y \le b \qquad ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta$ if $r = f(\theta), \;\; a \le \theta \le b$

With surface area you may have to substitute in for the x or y depending on your choice of ds to match the differential in the ds. With parametric and polar you will always need to substitute.

Improper Integral

An improper integral is an integral with one or more infinite limits and/or discontinuous integrands. Integral is called **convergent** if the limit exists and has a finite value and **divergent** if the limit doesn't exist or has infinite value.

Infinite Limit

1. $\int_a^\infty f(x) \, dx = \lim_{t \to \infty} \int_a^t f(x) \, dx$
2. $\int_{-\infty}^b f(x) \, dx = \lim_{t \to -\infty} \int_t^b f(x) \, dx$
3. $\int_{-\infty}^\infty f(x) \, dx = \int_{-\infty}^c f(x) \, dx + \int_c^\infty f(x) \, dx$ provided **both** integrals are convergent.

Discontinuous Integrand

1. Discontinuity at a : $\int_a^b f(x) \, dx = \lim_{t \to a^+} \int_t^b f(x) \, dx$
2. Discontinuity at b : $\int_a^b f(x) \, dx = \lim_{t \to b^-} \int_a^t f(x) \, dx$
3. Discontinuity at $a < c < b$: $\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx$ provided **both** are convergent.

Comparison Test for Improper Integrals : If $f(x) \ge g(x) \ge 0$ on $[a, \infty)$ then,

1. If $\int_a^\infty f(x) \, dx$ is convergent then $\int_a^\infty g(x) \, dx$ is convergent (if larger converges so does the smaller).
2. If $\int_a^\infty g(x) \, dx$ is divergent then $\int_a^\infty f(x) \, dx$ is divergent (if smaller diverges so does the larger).

Useful fact : If $a > 0$ then $\int_a^\infty \frac{1}{x^p} \, dx$ converges if $p > 1$ and diverges for $p \le 1$.

Approximating Definite Integrals

For given integral $\int_a^b f(x) \, dx$ and n (must be even for Simpson's Rule) define $\Delta x = \frac{b-a}{n}$ and divide $[a, b]$ into n subintervals $[x_0, x_1]$, $[x_1, x_2]$, …, $[x_{n-1}, x_n]$ with $x_0 = a$ and $x_n = b$ then,

Midpoint Rule : $\int_a^b f(x) \, dx \approx \Delta x \left[f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*) \right]$, x_i^* is midpoint $[x_{i-1}, x_i]$

Trapezoid Rule : $\int_a^b f(x) \, dx \approx \frac{\Delta x}{2} \left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n) \right]$

Simpson's Rule : $\int_a^b f(x) \, dx \approx \frac{\Delta x}{3} \left[f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n) \right]$